Proof. The first equivalence is a straight consequence of the fact that a relation satisfies $\text{unique}(X)$ if and only if it satisfies the FD $X \rightarrow R$. For the second equivalence we first show that if $\Sigma \models_{R_s} X \rightarrow R$ and $X \subseteq R_s$ hold, then $\Sigma \models_{R_s} \text{Codd}(X)$ holds as well. In fact, let $r$ denote an arbitrary relation over $R$ that satisfies $\Sigma$ and $R_s$. It follows that $r$ satisfies $X \rightarrow R$ as well since $\Sigma \models_{R_s} X \rightarrow R$. Since $X \subseteq R_s$ holds, we know that $r$ satisfies the NFS $X$. Consequently, $r$ satisfies $\text{Codd}(X)$.

It remains to show that if $\Sigma \not\models_{R_s} X \rightarrow R$ or $X \not\subseteq R_s$, then $\Sigma \not\models_{R_s} \text{Codd}(X)$.

Suppose first that $\Sigma \not\models_{R_s} X \rightarrow R$. Then there is some relation $r$ over $R$ that satisfies $\Sigma$ and the NFS $R_s$, but violates the FD $X \rightarrow R$. Consequently, there are two tuples $t,t' \in r$ such that $t[X] = t'[X]$, and $t$ and $t'$ are $X$-total and $t' \neq t'$. The two-tuple relation $\{t,t'\}$ shows that $\Sigma \not\models_{R_s} \text{Codd}(X)$ since $r$ satisfies $\Sigma$ and $R_s$ (since $t' \not\subseteq r$ holds), but violates $\text{Codd}(X)$.

Suppose now that $X \not\subseteq R_s$ holds, i.e., $X - R_s$ is non-empty. In this case we define a single-tuple relation $r := \{t\}$ for some tuple $t$ over $R$ such that $t[A] := n_1$, for all $A \in R - R_s$ and $t[B] \in \text{dom}(B) - \{n_1\}$ for all $B \in R_s$. It follows that $r$ satisfies $\Sigma$ and the NFS $R_s$, but $r$ violates $\text{Codd}(X)$ since $X \not\subseteq R - R_s$.

Theorem 16 (Theorem 4 restated) Let $R$ be some relation schema, let $\Sigma$ be a set of standard FDs and let $R_s$ be an NFS over $R$. For all relations $r$ over $R$ it holds that $r$ is an Armstrong table for $\Sigma$ and $R_s$ if and only if both of the following conditions are satisfied:

1. for all non-empty $X \subseteq R$ we have
   
   $$X^*_{\Sigma,R_s} = \bigcap \{w(Z) \mid X \subseteq Z \in \text{ag}^g(r)\},$$

2. total$(r) = R_s$.

Proof Sufficiency. Let $r$ be a relation over $R$ that satisfies the conditions. We show that $r$ is an Armstrong table for $\Sigma$ and $R_s$.

Let $X \rightarrow A \in \Sigma$. That is, $A \in X^*_{\Sigma,R_s}$. Assume that there are distinct tuples $t,t' \in r$ such that $t[X] = t'[X]$ and $t,t'$ are $X$-total. That is, $X \subseteq X' = \text{ag}^g(t,t')$. Hence, the first condition shows that $A \in w(X')$, and thus $A \in \text{ag}^g(t,t')$. Therefore, $t[A] = t'[A]$ holds. We have shown that $r$ satisfies $\Sigma$.

Let $X \rightarrow A \notin X^*_{\Sigma,R_s}$. Hence, $A \notin X^*_{\Sigma,R_s}$. By the first condition there is some $Z \in \text{ag}^g(r)$ such that $X \subseteq Z$ and $A \notin w(Z)$. In particular, there are tuples $t,t'$ such that $X \subseteq Z = \text{ag}^g(t,t')$ and $A \notin \text{ag}^g(t,t')$. That is, we have $t[X] = t'[X]$, and $t,t'$ are $X$-total and $t[A] \neq t'[A]$. This shows that $r$ violates every FD not in $\Sigma_{R_s}$.

Finally, the condition total$(r) = R_s$ ensures that $r$ satisfies every NFS implied by $R_s$ and violates every NFS not implied by $R_s$. Consequently, $r$ is indeed an Armstrong table for $\Sigma$ and $R_s$.

Necessity. Let $r$ be a relation over $R$ that is Armstrong for $\Sigma$ and $R_s$. We show that $r$ satisfies the conditions.

Let $t,t' \in r$ be distinct tuples such that $X \subseteq X' = \text{ag}^g(t,t')$. As $r$ satisfies $\Sigma_{R_s}$, we have $X^*_{\Sigma,R_s} \subseteq \text{ag}^g(t,t')$, and thus $X^*_{\Sigma,R_s} \subseteq w(X')$. Therefore, $X^*_{\Sigma,R_s} \subseteq \bigcap \{w(Z) \mid X \subseteq Z \in \text{ag}^g(r)\}$.

Next we show that $X^*_{\Sigma,R_s} \subseteq \bigcap \{w(Z) \mid X \subseteq Z \in \text{ag}^g(r)\}$. Assume there is an $A \notin X^*_{\Sigma,R_s}$ such that $A \in \{w(Z) \mid X \subseteq Z \in \text{ag}^g(r)\}$. Then we have $A \in \text{ag}^g(t,t')$ for all distinct tuples $t,t' \in r$ with $X \subseteq Z = \text{ag}^g(t,t')$. That is, $r$ satisfies $X \rightarrow A$. This, however, contradicts the assumption $A \notin X^*_{\Sigma,R_s}$, since $r$ is Armstrong for $\Sigma$ and $R_s$. Consequently, $X^*_{\Sigma,R_s} \subseteq \bigcap \{w(Z) \mid X \subseteq Z \in \text{ag}^g(r)\}$.

Finally, since $r$ is Armstrong for $\Sigma$ and $R_s$ it follows that total$(r) = R_s$.

Theorem 17 (Theorem 5 restated) Let $R$ be some relation schema, let $\Sigma$ be a set of standard FDs and let $R_s$ be an NFS over $R$. For all relations $r$ over $R$ it holds that $r$ is an Armstrong table for $\Sigma$ and $R_s$ if and only if the following conditions are satisfied:

1. $\forall A \in R \forall X \in \text{max}_{\Sigma,R_s}(A)(X \in \text{ag}^g(r) \land A \notin w(X))$, 
2. $\forall X \in \text{ag}^g(r)(X^*_{\Sigma,R_s} \subseteq w(X))$, and
3. total$(r) = R_s$.

Proof Sufficiency. Let $r$ be some relation over $R$ that satisfies conditions 1, 2. and 3. We show that $r$ is an Armstrong table for $\Sigma$ and $R_s$.

Let $X \rightarrow A \in \Sigma$. Assume that there are distinct tuples $t,t' \in r$ such that $t[X] = t'[X]$ and $t$ is $X$-total. That is, $X \subseteq X' = \text{ag}^g(t,t')$. Note that $A \in (X')^*_{\Sigma,R_s}$ by soundness of the augmentation rule. Hence, condition 2. implies that $A \in w(X')$. In particular, $A \in \text{ag}^g(t,t')$. Therefore, $t[A] = t'[A]$. Hence, $r$ satisfies $\Sigma$.

Let $X \rightarrow A \notin X^*_{\Sigma,R_s}$. It follows that there is some $X' \in \text{max}_{\Sigma,R_s}(A)$ such that $X \subseteq X'$ and $A \notin (X')^*_{\Sigma,R_s}$. Condition 1. implies that $X' \in \text{ag}^g(r)$ and $A \notin w(X')$. Hence, there is some $Y \in \text{ag}^g(r)$ such that $(X',Y) \in \text{ag}(r)$ and $A \notin Y$. This shows that there are two distinct tuples $t,t' \in r$ such that $t[X'] = t'[X']$ and $t,t'$ are $X'$-total and $t[A] \neq t'[A]$. We have shown that $r$ violates every functional dependency that is not in $\Sigma_{R_s}$.
Condition 3. ensures that r satisfies every NFs implied by R, and violates every NFs not implied by R. Consequently, r is an Armstrong table for Σ and R_s.

Necessity. Let r be some relation over R that is an Armstrong table for Σ and R_s. We show that r satisfies conditions 1, 2, and 3.

Let A ∈ R, and let X ∈ maxΣ,R_s(A). That is, ΣR_s X → A and for all B ∈ R - X it is true that ΣR_s B → A. Since r is an Armstrong table for Σ and R_s it follows that r violates X → A and for all B ∈ R - X that r satisfies the FD X → A. The violation of X → A implies that there are distinct t, t' ∈ r such that X ⊆ ag^t(t, t') and A /∈ ag^t(t, t'). If there was some attribute C of R in ag^t(t, t') - X, then r would violate the FD XC → A. Consequently, X = ag^t(t, t').

We have just shown that for every A ∈ R and for every X ∈ maxΣ,R_s(A) it is true that X = ag^t(r) and A /∈ w(X), i.e., condition 1, holds.

Next we show that r satisfies condition 2. Therefore, X = ag^t(r).

Necessity. Let r be some relation over R that is an Armstrong table for Σ and R_s. We have just shown that for every A ∈ R and for every X ∈ maxΣ,R_s(A) it is true that X = ag^t(r) and A /∈ w(X), i.e., condition 1, holds.

We show that r satisfies condition 2. Therefore, X = ag^t(r).

Theorem 18 (Theorem 6 restated) Let R be some relation schema, let Σ be an FD set, and R_s an NFS over R. For all relations r over R it holds that r satisfies Σ and R_s if and only if r ≤ total(r) and for all A ∈ ag^t(r)(X ⊆ w(X)) holds.

Proof Sufficiency. The proof of Theorem 5 shows that r satisfies Σ, if ∀X ∈ ag^t(r)(X ⊆ w(X)) holds. Furthermore, if r ≤ total(r), then r is R_s-total.

Necessity. If r satisfies R_s, then r is R_s-total. It remains to show that ∀X ∈ ag^t(r)(X ⊆ w(X)) holds, if r satisfies Σ and R_s. Assume there is some X ∈ ag^t(r) and there is some A ∈ R such that R_s X → A. Since A /∈ w(X), there is some Y ⊆ R such that (X, Y) ∈ ag^t(r) and A /∈ Y. Consequently, there are distinct t, t' ∈ r such that X = ag^t(t, t') and Y = ag^t(t, t'). That is, t[X] = t'[X] and t, t' are X-total, and t[A] /∈ t'[A]. Hence, r violates the FD X → A. From A ∈ X ⊆ R_s, it follows that X → A ∈ ΣR_s. The definition of implication shows that r violates Σ.

Theorem 19 (Theorem 7 restated) Let R be a relation schema, R_s a null-free sub-schema over R, and Σ = Σ' ∪ (X → A) a set of standard functional dependencies over R. For C ⊆ R let V ∈ maxΣ,R_s(C). Then V ∈ maxΣ,R_s(C) or (C = A or A ∈ R_s) holds and there is some B ∈ X - V such that

i) VB ∈ maxΣ,R_s(C), if X ⊆ R_s, or

ii) V = W ∩ Z for some W ∈ maxΣ,R_s(C) and some Z ∈ maxΣ,R_s(B).

The proof will make use of the following simple observation:

Remark 4 Let Σ = Σ' ∪ (X → A) and X ⊆ R. When A ∈ U^Σ,R_s, we have U^Σ,R_s = \((U^A \cup R_s) \cup (U^B \cup R_s)\), if A ∈ R_s, otherwise, while U^Σ,R_s = U^Σ,R_s holds when A /∈ U^Σ,R_s. Furthermore, the following statements are equivalent: a) U^Σ,R_s ⊆ U^Σ,R_s, b) A ∈ U^Σ,R_s - U^Σ,R_s, and c) A /∈ U^Σ,R_s, X ⊆ UR_s ∩ U^Σ,R_s, cf. Algorithm 1.

Proof From V ∈ maxΣ,R_s(C) and V ⊆ V^Σ,R_s ⊆ V^Σ,R_s, we obtain C /∈ V^Σ,R_s and C /∈ V^Σ,R_s, if V ∈ maxΣ,R_s(C) we are done. Otherwise, there is some W ∈ maxΣ,R_s(C) with V ⊆ W. By V ∈ maxΣ,R_s(C) we obtain C ∈ W^Σ,R_s ∩ W^Σ,R_s, so that W^Σ,R_s ⊆ W^Σ,R_s. When applying Remark 4 to the set W, we note A ∈ W^Σ,R_s - W^Σ,R_s and, therefore, (C = A or A ∈ R_s).

Remark 4 for W further yields A /∈ V^Σ,R_s, and X ⊆ WR_s ∩ W^Σ,R_s. Hence, A /∈ V^Σ,R_s, as V ⊆ W. Assume A ∈ V^Σ,R_s - V^Σ,R_s. Then, C /∈ A as C /∈ V^Σ,R_s, and thus A ∈ R_s. When applying Remark 4 to the set V, we observe V^Σ,R_s = (V^A \cup R_s) \cup (V^B \cup R_s). Hence, C /∈ (V^A \cup R_s). By V ∈ maxΣ,R_s(C) we obtain A ∈ V. Thus A ∈ V ⊆ W ∩ W^Σ,R_s, which contradicts A ∈ W^Σ,R_s. Hence, our assumption is false, and A /∈ V^Σ,R_s holds. From A /∈ V^Σ,R_s and Remark 4 for V we conclude V^Σ,R_s = V^Σ,R_s, and X ⊆ V_R_s ∩ V^Σ,R_s, that is, (X ⊆ VR_s or X ⊆ V^Σ,R_s). By X ⊆ WR_s we obtain ((X - R_s) ∩ (W - V)) /∈ Σ or X ⊆ V^Σ,R_s. Therefore, V is a subset of a member U of

\[ V := \{W - B : B \in X - R_s\} \cup \{W \cap Z : W \subseteq Z \subseteq \bigcup_{B \in X} \text{maxΣ,R_s(B)}\}. \]

From U ⊆ W and W ∈ maxΣ,R_s(C) we get C /∈ U^Σ,R_s. By definition of V, we observe X ⊆ VR_s ∩ U^Σ,R_s which yields U^Σ,R_s = U^Σ,R_s by Remark 4. Thus, we obtain C /∈ U^Σ,R_s. From V ∈ maxΣ,R_s(C), we derive V = U, that is, V itself is a member of V. This concludes the proof of Theorem 7.

Theorem 20 (Theorem 11 restated) Algorithm 10, on input \((R, \Sigma, R_s)\), computes an Armstrong table for Σ and R_s.
Proof Let $r$ denote the output of Algorithm 10. We show first that the output $r$ of Algorithm 10 is a subsumption-free relation. Let $t,t' \in r$ denote two distinct tuples. Suppose that $t$ results from some $X \in \text{max}(A)$ and $t'$ results from some $Y \in \text{max}(A)$. The construction guarantees that $t[A] \neq t'[A]$ and $t[A] \neq n_i \neq t'[A]$. Hence, neither of $t,t'$ subsumes the other. Suppose that $t$ results from some $X \in \text{max}(A)$ and $t'$ results from some $Y \in \text{max}(B)$ where $A \neq B$. If $t'[A] \neq n_i$, then $t[A] \neq t'[A]$ and $t[A] \neq n_i \neq t'[A]$. If $t[B] \neq n_i$, then $t[B] \neq t'[B]$ and $t[B] \neq n_i \neq t'[B]$. If $t'[A] = n_i$ and $t[B] = n_i$, then neither of $t,t'$ subsumes the other. Finally, if $t$ results from some $X \in \text{max}(A)$ and $t'$ results from step (A7), or if $t$ and $t'$ result from step (A7), then the construction (distinct values) guarantees that neither of $t,t'$ subsumes the other. Hence, $r$ is a subsumption-free relation.

We show $r$ to be an Armstrong table for $\Sigma$ and $R_s$.

Let $X \rightarrow A \in \Sigma$. Assume that $r$ violates $X \rightarrow A$. Then there are distinct $t,t' \in r$ such that $t[X] = t'[X]$ and $t,A \neq t'[A]$. Since $X \neq \emptyset$, it follows from construction that $(t,t') = \{t_2,1,t_2,2\}$ for some positive integer $i$. According to the steps (A3) and (A4) we conclude that there is some $X' \in \text{max}(R)$ and there is some $B \in Z := \{C \in R \mid X' \in \text{max}(C)\}$ such that $X \subseteq X'$ and $A \in Z_{R_s}$. Suppose that $A \in Z$. Consequently, $X \subseteq X' \in \text{max}(C)$, i.e., $A \notin (X')_{\Sigma,R_s}$. However, due to the soundness of the augmentation rule we conclude that $A \notin X'_{\Sigma,R_s}$. This means that $X \rightarrow A \notin \Sigma_{R_s}^*$, which contradicts $X \rightarrow A \in \Sigma$. Suppose now that $A \in R_s - Z$. From $X' \in \text{max}(B)$ it follows that $X' \rightarrow B \in \Sigma_{R_s}^*$ and that $X'A \rightarrow B \in \Sigma_{R_s}^*$. From $X \rightarrow A \in \Sigma_{R_s}^*$ follows $X' \rightarrow A \in \Sigma_{R_s}^*$ by the soundness of the augmentation rule. From the soundness of the reflexivity axiom and the union rule we conclude that $X' \rightarrow X'A \in \Sigma_{R_s}^*$. An application of the null transitivity rule to $X' \rightarrow X'A$, $X'A \rightarrow B$ and $A \in R_s$ results in $X' \rightarrow B$. Due to the soundness of the null transitivity rule we conclude that $X' \rightarrow B \in \Sigma_{R_s}^*$. This contradicts the fact that $X' \in \text{max}(B)$. We have just shown that $r$ satisfies $\Sigma$. The construction in steps (A3) and (A4) ensures that $r$ satisfies the NFS $R_s$.

It is not difficult to see that the relation $r$ violates all standard FDs $X \rightarrow A \notin \Sigma_{R_s}^*$. In fact, by definition of $\text{max}(A)$ there is some $X' \subseteq R$ such that $X' \in \text{max}(A)$ and $X \subseteq X'$. Step (A4) guarantees that there are some distinct $t,t' \in r$ such that $t[X'] = t'[X']$, $t,A \neq t'[A]$, $t,B \neq t'[B]$ and $t[A] \neq t'[A]$. Hence, $r$ violates $X \rightarrow A$.

It is now quite easy to see that the relation $r$ is an Armstrong table for $\Sigma$ and $R_s$. In fact, step (A7) guarantees that $r$ is total on precisely those attributes of $R$ that belong to $R_s$. Note that $R_s \subseteq \text{total}(r)$ always holds due to the construction. Hence, if $\text{total}(r) - R_s \neq \emptyset$, then we need to add some tuples with occurrences of $n_i$ in all columns $A \in \text{total}(r) - R_s$. If $R_s = \emptyset$, $\text{total}(r) = R$ and $|R| > 1$, then we require two tuples to ensure that $r$ remains subsumption-free. Otherwise, we can just add a single tuple with occurrences of $n_i$ in all columns $A \in \text{total}(r) - R_s$. Hence, $r$ satisfies precisely those null-free subschema constraints implied by $R_s$ (namely the subsets of $R_s$).

\[ \frac{1 + 8 \cdot \text{max}_{\Sigma,R_s}(R)}{2} \leq |r| \leq 2 \times \text{max}_{\Sigma,R_s}(R) + 1. \]

Proposition 5 (Proposition 1 restated) Let $\Sigma$ be a set of standard FDs, let $R_s$ be some NFS over some relation schema $R$, and let $r$ be an Armstrong table for $\Sigma$ and $R_s$. Then $|\text{max}_{\Sigma,R_s}(R)| \leq |ag(r)| \leq \binom{|\Sigma|}{1}$.

Proof The first condition of Theorem 5 implies that $|\text{max}_{\Sigma,R_s}(R)| \leq |ag(r)|$. Moreover, $|ag^*(r)| \leq |ag(r)|$, and $|ag(r)| \leq \binom{|\Sigma|}{1}$ since every distinct pair of distinct tuples in $r$ has precisely one agree set.

Proposition 6 (Proposition 2 restated) The complexity of finding an Armstrong table, given a set of standard functional dependencies and a null-free subschema, is precisely exponential in the number of attributes.

Proof The time complexity of Algorithm 10 is dominated by that of Algorithm 8 which runs clearly in time exponential in the number of attributes.

It remains to show that there is a set $\Sigma$ of standard FDs and an NFS $R_s$ for which the number of tuples in each Armstrong table for $\Sigma$ and $R_s$ is exponential in the number of attributes. According to Proposition 1 it suffices to find a set $\Sigma$ of standard FDs such that $\text{max}_{\Sigma,R_s}(R)$ is exponential in the number of attributes. Such a set $\Sigma$ is given by

$$\bigcup_{1 \leq i \leq n} \{A_{2i-1},A_{2i} \rightarrow B\}$$

and the NFS $R_s = A_1 \cdots A_n B$. This is the same set that Beeri, Dowd, Fagin and Statman used to show that the time complexity of finding an Armstrong relation for FDs over total relations takes at least exponential time in the number of attributes [9]. This set works here for the same purpose since all FDs in $\Sigma$ have the same right-hand side.

Proposition 7 (Proposition 3 restated) Let $\Sigma$ be a set of standard FDs, let $R_s$ be some NFS over some relation schema $R$, and let $r$ be a minimum-sized Armstrong table for $\Sigma$ and $R_s$. Then

$$\sqrt{1 + 8 \cdot \text{max}_{\Sigma,R_s}(R)} \leq |r| \leq 2 \times \text{max}_{\Sigma,R_s}(R) + 1.$$
Theorem 22 (Theorem 13 restated) There is some
Proof The lower bound follows from Proposition 1. In-
deed, it follows that $|\max_{\Sigma,R}(R)| \leq \left(\frac{1}{2}\right)^{|\Sigma|}$. Consequently,
we have that $2 \times |\max_{\Sigma,R}(R)| + 2 \leq |r|$. The upper
bound $2 \times |\max_{\Sigma,R}(R)| + 2$ follows immediately from
Theorem 11. However, Algorithm 10 outputs an Arm-
strong table of size $2 \times |\max_{\Sigma,R}(R)| + 2$ if and only if
$\text{total}(r) = R$ holds before step (A7) and $R_s = \emptyset$. We
have $B \in \text{total}(r)$ before step (A7) if and only if $B \in XA \cup R_s$ for every maximal set $X \in \max_{\Sigma,R}(A)$ and all $A \in R$. Therefore, we have $\text{total}(r) = R$ before step
(A7) and $R_s = \emptyset$ if and only if $\max_{\Sigma,R}(A) = \{R - A\}$
for all $A \in R$ and $R_s = \emptyset$. This again holds if and
only if $\Sigma = \emptyset$ and $R_s = \emptyset$. In this special case and
when $|r| > 1$, one may use the Algorithm from Man-
nila and Räihä [98] to compute an Armstrong relation
for $\Sigma$ of size $\max_{\Sigma,R}(R) + 1 = |R| + 1$ and add two
tuples according to step (A7) of Algorithm 10 to obtain
an Armstrong table for $\Sigma$ and $R_s$. We show
that its size at least $2 \times |\max_{\Sigma,R}(R)|$. By Theo-
rem 5 there are tuples $t_i, t_i' \in r$ for all $i = 1, \ldots, n$ such that $ag(t_i, t_i') = R - A_{i-1}A_i$ and $t_i[A_i] \neq t_i'[A_i]$. We
conclude $t_i[A_{i+1}] = ni = t_i'[A_{i+1}]$ for $i = 1, \ldots, n$ since $r$ satisfies the FD $A_{i+1} \rightarrow A_i$, and also $t_i[A_i] \neq t_i'[A_i]$ for $i = 1, \ldots, n$. Obviously, the tuples $t_i, t_i', t_n, t_n'$
are mutually distinct, so that $r$ has size at least $2 \times n = 2 \times |\max_{\Sigma,R}(R)|$.

We will show first that $\Sigma$ is non-redundant (no sub-
set of $\Sigma$ implies all FDs in $\Sigma$), and then show that $\Sigma$ is
an optimal cover of itself. We note that for every FD $\sigma \in \Sigma$, where $X = \text{LHS}(\sigma)$ denotes the attribute set on the
left-hand side of $\sigma$, the closure $X'_{\Sigma\setminus\{\sigma\},R_s}$ of $X$ with
respect to $\Sigma \setminus \{\sigma\}$ and $R_s$ is itself, i.e., $X'_{\Sigma\setminus\{\sigma\},R_s} = X$. The reason is that there is no $\sigma' \in \Sigma \setminus \{\sigma\}$ such
that $\text{LHS}(\sigma') \subseteq X$. Hence, $\Sigma \notin X'_{\Sigma\setminus\{\sigma\},R_s}$ and we
conclude that $\sigma$ is not implied by $\Sigma \setminus \{\sigma\}$ and $R_s$. That
is, $\Sigma$ is non-redundant.

Next we remark that every optimal cover $\Sigma'$ of $\Sigma$ with
respect to $R_s$ contains only FDs $X \rightarrow Y$ such that
$Y = C$. Suppose, to the contrary, that there is some
FD $X \rightarrow Y$ in $\Sigma'$ such that $Y \neq C$. If $Y-X \neq \emptyset$
and $Y \neq C$, then $\Sigma'$ is not optimal since
$X \rightarrow Y \quad \text{is equivalent to } \Sigma \text{ but contains less attributes than } \Sigma'$.
If $Y-X = \emptyset$, then $\Sigma' \setminus \{X \rightarrow Y\}$ is equivalent to $\Sigma$ but
contains less symbol occurrences than $\Sigma'$. If $Y-X \neq \emptyset$
and $Y \neq C$, then $\Sigma \notin \Sigma'$, $X \rightarrow Y$, and, therefore,
$\Sigma'$ is not a cover of $\Sigma$ with respect to $R_s$. Moreover,
every FD $X \rightarrow Y$ in an optimal cover $\Sigma'$ of $\Sigma$ with
respect to $R_s$ satisfies that $C \notin X$. If there was an FD
$X \rightarrow C \in \Sigma'$ and $C \in X$, then
$X \rightarrow C \quad \text{is equivalent to } \Sigma \text{ but contains less attributes than } \Sigma'$.

Next we prove that there is no cover $\Sigma'$ of $\Sigma$ with
respect to $R_s$ with a smaller number of attribute occur-
rences. Suppose there were an optimal cover $\Sigma'$ of $\Sigma$
with respect to $R_s$ with fewer number of attribute occurrences
than $\Sigma$. Then for all $\sigma' \in \Sigma'$ it is the case that $\Sigma \models \sigma'$. Consequently, there must be some
$\sigma \in \Sigma$ such that $\text{LHS}(\sigma) \subseteq \text{LHS}(\sigma')$. Suppose every
FD $\sigma \in \Sigma$ has the property that $\text{LHS}(\sigma) \subseteq \text{LHS}(\sigma')$
for a different FD $\sigma' \in \Sigma'$. Then $\Sigma'$ contains at least as
many attribute occurrences as $\Sigma$, a contradiction. Oth-
wise, there is a proper subset $\Sigma''$ of $\Sigma$ such that every
FD $\sigma' \in \Sigma''$ has the property that $\text{LHS}(\sigma) \subseteq \text{LHS}(\sigma')$
for some $\sigma \in \Sigma''$. Consequently, $\Sigma''$ implies every FD in
$\Sigma'$ with respect to $R_s$ and therefore also every FD in $\Sigma$.
This, however, is impossible since $\Sigma$ is non-redundant.

Thus we have just shown that $\Sigma$ is its own opti-
mal cover with respect to $R_s$, and thus exponential in
the number of attributes. Now we show that there is
an Armstrong table for $\Sigma$ and $R_s$ where the number of tuples is in $O(n)$. It suffices to show that the set
$\max_{\Sigma,R}(C)$ contains a number of elements that is lin-
ear in the number of attributes. For each $i = 1, \ldots, n$ we
have $\max_{\Sigma,R}(A_i) = R - A_i$, and $\max_{\Sigma,R}(B_i) = R - B_i$. These are $2n$ different maximal sets in total. The set $\max_{\Sigma,R}(C)$ consists of the following $n$ elements:
\( R = A_i B_i C, i = 1, \ldots, n. \) Therefore, \( \max_{\Sigma, R_i}(R) \) has \( 3n \) different elements. Using Mannila and Räihä’s algorithm [98] (which applies since we are in the case where \( R_s = R \)) we can easily create an Armstrong table for \( \Sigma \) and \( R_s \) that has \( 3n + 1 \) tuples only. \( \square \)