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On a problem of Fagin concerning multivalued dependencies in relational databases

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Abstract

Multivalued dependencies (MVDs) are an important class of relational constraints that is fundamental to relational database design. Reflexivity axiom, complementation rule, and pseudo-transitivity rule form a minimal set of inference rules for the implication of MVDs. The complementation rule plays a distinctive role as it takes into account the underlying relation schema R which the MVDs are defined on. The R-axiom $\emptyset \twoheadrightarrow R$ is much weaker than the complementation rule, but is sufficient to form a minimal set of inference rules together with augmentation and pseudo-difference rule. Fagin has asked whether it is possible to reduce the power of the complementation rule and drop the augmentation rule at the same time and still obtain a complete set. It was argued that there is a trade-off between complementation rule and augmentation rule, and one can only dispense with one of these rules at the same time. It is shown in this paper that an affirmative answer to Fagin's problem can nevertheless be achieved. In fact, it is proven that R-axiom together with a weaker form of the reflexivity axiom, pseudo-transitivity rule and exactly one of union, intersection or difference rule form such desirable minimal sets. The positive solution to this problem gives further insight into the difference between the notions of functional and multivalued dependencies.

Keywords: Multivalued dependency; Complete axiomatisation; Inference rule; Minimality; Functional dependency; Relational database; Database; Semantics of data

1. Introduction

Relational databases still form the core of most database management systems after more than three decades after their introduction in [11]. The relational data model organises data into a collection of relations. As the semantics of data cannot solely be captured by structures, different classes of dependencies can be utilised for improving the representation of the target database. Dependencies restrict the set of possible instances of a database schema to those which are considered meaningful for the application in mind. Excellent surveys on relational dependencies can be found in [15,27].

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Functional dependencies (FDs) between sets of attributes have always played a central role in the study of relational databases [11,12,5,7,8], and seem to be central for the study of database design in other data models as well [1,18,20,22,26,30,31]. The notion of a functional dependency is well-understood and the semantic interaction between these dependencies has been syntactically captured by Armstrong's well-known axioms [2,3]. However, FDs are incapable of modelling many important properties that database users have in mind. Multivalued dependencies (MVDs) provide a more general notion and offer a response to the shortcomings of FDs [13,14,32]. A relation exhibits an MVD precisely when it is decomposable into two of its projections without loss of information [14]. This property is fundamental to relational database design, in particular 4NF [14], and a lot of research has therefore been devoted to studying the behaviour of these dependencies. Recently, extensions of multivalued dependencies have been found very useful for various design problems in advanced data models such as the nested relational data model [16], the entity-relationship model [28], data models that support nested lists [19] and XML [29].

Beeri et al. [6] were the first to give a complete axiomatisation of MVDs . That means a set of inference rules was provided which allows to infer syntactically all those dependencies that are logically implied by the set of dependencies that a database designer chooses to specify. This gives the designer complete knowledge about all the logical consequences of the set of dependencies, and therefore helps to avoid inconsistencies and undesired behaviour in the target database. The notion of an MVD is more difficult to grasp than that of an FD. It is therefore interesting to study which of the inference rules are really essential to capture the implication of multivalued dependencies. Any rule that does not allow to infer any other dependencies than those already inferred by the rest of the inference rules is redundant, and therefore only blurs the true character of the notion of an MVD. The first minimal axiomatisation of multivalued dependencies was obtained by Mendelzon [23], i.e., a set of inference rules in which none of the rules can be omitted without losing completeness. This set consists of reflexivity axiom, complementation and pseudo-transitivity rule. The complementation rule strongly reflects the dependence of the notion of a multivalued dependency from the underlying relation schema, and does not have any analogue in the axiomatisation of FDs in which all rules apply independently of whatever relation schema the attributes are embedded in. Biskup [9] replaces the complementation rule by a much weaker axiom which still reflects the dependence from the underlying relation schema. In order to regain a completeness of the inference rules the more powerful augmentation rule replaced the reflexivity axiom. Fagin asked in a personal communication with Biskup [9] whether it is possible to reduce the power of the complementation rule and drop the augmentation rule at the same time and still obtain a complete set. Biskup [9] argued that there is a trade-off between complementation rule and augmentation rule, and one can only dispense with one of these rules at the same time.

It is shown in this paper that Fagin's question has an affirmative solution if one considers the union or one of the decomposition rules. In fact, three minimal axiomatisations are proposed in each of which the power of both complementation rule and augmentation rule is strongly reduced. Apart from the fact that these axiomatisations provide an answer to Fagin's question, they also represent new simple syntactic descriptions of the semantic interaction of multivalued dependencies. Moreover, the results complement Biskup's previous answer to Fagin's question [9] in which union and decomposition rules were not considered. One may also argue that the appearance of the union or decomposition rule in a minimal set of inference rules is rather natural since these three rules are essential to the existence of what is known in the literature as the dependency basis [4,17]. One of our solutions to Fagin's question allows to characterise syntactically the difference between the notions of functional and multivalued dependencies. That is, MVDs require the R-axiom, by which the dependence of the notion of an MVD on the underlying relation schema R is reflected, and while general transitivity holds for functional dependencies only pseudo-transitivity is valid for MVDs. It has been stated (e.g. [24,25]) in a number of practitioner reports that MVDs are difficult to learn, teach, model and handle. On the other hand MVDs are essential to the database normalisation process and are therefore taught in most database courses and introduced in most database books. Our findings may help to comprehend and convey the concept of a multivalued dependency easier and faster, and invite database practitioners and modellers to utilise them more. Finally, our result may simplify the quest of finding suitable and comprehensible extensions for the notion of an MVD in any advanced data models.

The article is organised as follows. Section 2 is used to define the fundamental concepts used in this paper, and to summarise relevant previous research on axiomatisations of multivalued dependencies. The solutions to Fagin's problem are proposed in Section 3 where the completeness and minimality of each of the axiomatisations are formally proven. A brief discussion of the solutions and a description of the difference between the notions of FDs and MVDs are given in Section 4. Moreover, we propose axiomatisations of FDs together with MVDs in which the power of the complementation rule is reduced and the augmentation rule is dropped at the same time. We conclude in Section 5 with some comments on future work.

2. Definitions

A *relation schema* is a finite set $R = \{A_1, \dots, A_n\}$ of distinct symbols, called *attributes*, which represent column names of a relation. Each attribute A_i of a relation schema has an infinite domain $dom(A_i)$ which represents the set of possible values that can occur in the column named A_i. If X and Y are sets of attributes, then we may write XY for $X \cup Y$. If $X = \{A_1, \dots, A_m\}$, then we may write $A_1 \cdots A_m$ for X. In particular, we may write simply A to represent the singleton $\{A\}$. A *tuple* over $R = \{A_1, \dots, A_n\}$ (R-tuple or simply tuple, if R is understood) is a function $t : R \to \bigcup_{i=1}^{n} dom(A_i)$ with $t(A_i) \in dom(A_i)$ for $i = 1, \dots, n$. For $X \subseteq R$ let $t[X]$ denote the restriction of the tuple t over R on X, and $Dom(X) = \prod_{A \in X} dom(A)$ the Cartesian product of the domains of attributes in X. A *relation* r (over R) is a set of tuples over R. Let $r[X] = \{t[X] \mid t \in r\}$ denote the *projection* of the relation r over R on $X \subseteq R$. For $X, Y \subseteq R$, $r_1 \subseteq Dom(X)$ and $r_2 \subseteq Dom(Y)$ let $r_1 \bowtie r_2 = \{t \in Dom(XY) \mid \exists t_1 \in r_1, t_2 \in r_2 \text{ with } t[X] = t_1[X] \text{ and } t[Y] = t_2[Y]\}$ denote the *natural join* of r_1 and r_2. Note that the 0-ary relation $\{()\}$ is the projection $r[\emptyset]$ of r on $\emptyset$ as well as left and right identity of the natural join operator.

A *functional dependency* (FD) [12] on R is an expression $X \to Y$ where $X, Y \subseteq R$. A relation r over R *satisfies* the FD $X \to Y$, denoted by $\models_r X \to Y$, if and only if every pair of tuples in r that agrees on each of the attributes in X also agrees on the attributes in Y. That is, $\models_r X \to Y$ if and only if $t_1[Y] = t_2[Y]$ whenever $t_1[X] = t_2[X]$ holds for any $t_1, t_2 \in r$.

A *multivalued dependency* (MVD) [14,32] on R is an expression $X \twoheadrightarrow Y$ where $X, Y \subseteq R$. A relation r over R *satisfies* the MVD $X \twoheadrightarrow Y$, denoted by $\models_r X \twoheadrightarrow Y$, if and only if for all $t_1, t_2 \in r$ with $t_1[X] = t_2[X]$ there is some $t \in r$ with $t[XY] = t_1[XY]$ and $t[X(R - Y)] = t_2[X(R - Y)]$. Informally, the relation r satisfies $X \twoheadrightarrow Y$ when the value on X determines the set of values on Y independently from the set of values on $R - Y$. This actually suggests that the relation schema R is overloaded in the sense that it carries two independent facts XY and $X(R - Y)$. More precisely, Fagin [14] shows that MVDs "provide a necessary and sufficient condition for a relation to be decomposable into two of its projections without loss of information (in the sense that the original relation is guaranteed to be the join of the two projections)". This means that $\models_r X \twoheadrightarrow Y$ if and only if $r = r[XY] \bowtie r[X(R - Y)]$. This characteristic of MVDs is fundamental to relational database design and 4NF [14].

For the design of a relational database schema dependencies are normally specified as semantic constraints on the relations which are intended to be instances of the schema. During the design process one usually needs to determine further dependencies which are logically implied by the given ones. Let Σ denote a set of dependencies on R and σ a further dependency on R. We say that Σ (finitely) *implies* σ (σ is a consequence of Σ), denoted by $\Sigma \models_{(f)} \sigma$, if and only if every (finite) relation over R that satisfies all dependencies in Σ also satisfies σ. Thus, Σ (finitely) implies σ precisely if there is no (finite) "counterexample relation" that satisfies Σ but not σ. Since real life databases are inherently finite our attention should firstly be directed towards the finite implication. However, it is well-known that finite and general implication coincide for both FDs and MVDs [21]. A dependency is called *trivial* if it is valid, that is, a consequence of the empty set. It is straightforward to verify that an FD $X \to Y$ is trivial if and only if $Y \subseteq X$ holds; and an MVD $X \twoheadrightarrow Y$ is trivial if and only if $Y \subseteq X$ or $XY = R$. The *semantic hull* of Σ under implication is defined as $\Sigma^* = \{\sigma \mid \Sigma \models \sigma\}$, i.e., as the set of all dependencies implied by Σ. Next we consider syntactical inference rules for the implication of MVDs. The general form of these inference rules is

$$\frac{\text{premise}}{\text{conclusion}},$$

and inference rules without a premise are called axioms. In order to determine the semantic hull one can use the following set of inference rules for the implication of multivalued dependencies [6]. Note that we use the natural complementation rule [9] instead of the complementation rule that was originally proposed [6].

$$\frac{}{X \twoheadrightarrow Y}\ Y \subseteq X \qquad \frac{X \twoheadrightarrow Y}{XU \twoheadrightarrow YV}\ V \subseteq U \qquad \frac{X \twoheadrightarrow Y,\ Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

(reflexivity, $\mathcal{R}$) (augmentation, $\mathcal{A}$) (pseudo-transitivity, $\mathcal{T}$)

$$\frac{X \twoheadrightarrow Y}{X \twoheadrightarrow R - Y}$$

(complementation, $\mathcal{C}$)

$$\frac{X \twoheadrightarrow Y,\ X \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \qquad \frac{X \twoheadrightarrow Y,\ X \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \qquad \frac{X \twoheadrightarrow Y,\ X \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z}$$

(union, $\mathcal{U}$) (difference, $\mathcal{D}$) (intersection, $\mathcal{I}$)

Beeri et al. [6] prove that this set of inference rules is both sound and complete for the implication of MVDs. Let $\Sigma \vdash_{\mathfrak{R}} \sigma$ denote the inference of σ from a set Σ of dependencies with respect to the set $\mathfrak{S}$ of inference rules. Let $\Sigma^+_{\mathfrak{S}} = \{\sigma \mid \Sigma \vdash_{\mathfrak{S}} \sigma\}$ denote the *syntactic hull* of Σ under inference using only rules from $\mathfrak{S}$. The set $\mathfrak{S}$ is called *sound* for the implication of dependencies if and only if for every relation schema R and for every set Σ of dependencies on R we have $\Sigma^+_{\mathfrak{S}} \subseteq \Sigma^*$. The set $\mathfrak{S}$ is called *complete* for the implication of dependencies if and only if for every relation schema R and for every set Σ of dependencies on R we have $\Sigma^* \subseteq \Sigma^+_{\mathfrak{S}}$.

An interesting question is now whether all the rules of a certain set of rules are really necessary to capture the implication of dependencies or whether there are any interrelations among the rules of the set. More precisely, an inference rule $\mathfrak{R}$ is said to be *independent* from the set $\mathfrak{S}$ of inference rules if and only if there is some relation schema R, some set Σ of dependencies on R and some further dependency σ with $\sigma \notin \Sigma^+_{\mathfrak{S}}$, but $\sigma \in \Sigma^+_{\mathfrak{S} \cup \{\mathfrak{R}\}}$. A complete set $\mathfrak{S}$ for the implication of dependencies is called *minimal* for the implication of dependencies if and only if every inference rule $\mathfrak{R}$ in $\mathfrak{S}$ is independent from $\mathfrak{S} - \{\mathfrak{R}\}$. It was shown by Mendelzon [23] that

$$\frac{}{X \twoheadrightarrow Y}\ Y \subseteq X \qquad \frac{X \twoheadrightarrow Y}{X \twoheadrightarrow R - Y} \qquad \frac{X \twoheadrightarrow Y,\ Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

(reflexivity, $\mathcal{R}$) (complementation, $\mathcal{C}$) (pseudo-transitivity, $\mathcal{T}$)

form such a minimal set of inference rules for the implication of MVDs. The complementation rule $\mathcal{C}$ plays a special role as it is the only rule which depends on the underlying relation schema R. In the same paper, Mendelzon further motivates the study of the independence of inference rules and comments in more detail on the special role of the complementation rule. Let $\mathfrak{M}$ denote the set consisting of $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{T}$.

The definition of MVDs refers to an underlying relation schema R. This fact must be reflected by the need of some version of the complementation rule with regard to a complete set of inference rules. The question arises whether there is any weaker form of the complementation rule that still suffices for gaining completeness. Biskup [9] shows that the following complete set of inference rules

$$\frac{}{\emptyset \twoheadrightarrow R} \qquad \frac{X \twoheadrightarrow Y}{XU \twoheadrightarrow YV}\ V \subseteq U \qquad \frac{X \twoheadrightarrow Y,\ Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

(R-axiom, $\mathcal{C}.1$) (augmentation, $\mathcal{A}$) (pseudo-transitivity, $\mathcal{T}$)

is minimal for the implication of MVDs. Let $\mathfrak{B}$ denote the set that consists of $\mathcal{C}.1$, $\mathcal{A}$ and $\mathcal{T}$. The R-axiom is much weaker than the complementation rule itself. The loss of expressiveness by replacing the complementation rule $\mathcal{C}$ in $\mathfrak{M}$ by the R-axiom $\mathcal{C}.1$ in $\mathfrak{B}$ is compensated by using the more powerful augmentation rule $\mathcal{A}$ in $\mathfrak{B}$ instead of the reflexivity axiom $\mathcal{R}$ in $\mathfrak{M}$. Indeed, the following inference shows that the reflexivity axiom $\mathcal{R}$ is derivable from $\{\mathcal{C}.1, \mathcal{A}, \mathcal{T}\}$.

$$\mathcal{A}: \frac{\mathcal{T}: \dfrac{\mathcal{C}.1: \dfrac{}{\emptyset \twoheadrightarrow R} \qquad \mathcal{A}: \dfrac{\mathcal{C}.1: \dfrac{}{\emptyset \twoheadrightarrow R}}{R \twoheadrightarrow R}\ R \subseteq R}{\emptyset \twoheadrightarrow \emptyset}}{X \twoheadrightarrow Y}\ Y \subseteq X$$

Biskup [9] mentions a personal communication in which "Fagin asked whether one can reduce the power of the complementation rule and drop the augmentation rule at the same time and still get a complete system, or whether there is a trade-off between complementation rule and augmentation rule, that is we can dispense only with one of these rules at the same time." In the same paper, it was argued that a trade-off actually holds. This is certainly true if one considers the set $\{\mathcal{R}, \mathcal{C}.1, \mathcal{C}, \mathcal{A}, \mathcal{T}\}$ (see Corollaries 8 and 9 in [9, p. 304]). We will show in this paper that there is no trade-off if one considers union rule $\mathcal{U}$, intersection rule $\mathcal{I}$ or difference rule $\mathcal{D}$.

3. Three minimal axiomatisations

Consider $\mathfrak{B} = \{\mathcal{C}.1, \mathcal{A}, \mathcal{T}\}$. The simple idea is to replace the augmentation rule $\mathcal{A}$ by the reflexivity axiom $\mathcal{R}$ and investigate whether there are any further inference rules that allow to regain completeness. As it turns out any singleton of $\{\mathcal{U}, \mathcal{I}, \mathcal{D}\}$ suffices. It is even possible to replace the reflexivity axiom $\frac{}{X \twoheadrightarrow Y}\; Y \subseteq X$ by the weaker axiom $\frac{}{X \twoheadrightarrow A}\; A \in X$. The main result of this paper is the following.

Theorem 1. *The following inference rules*

$$\frac{}{\emptyset \twoheadrightarrow R} \quad (R\text{-axiom}, \mathcal{C}.1) \qquad \frac{}{X \twoheadrightarrow A}\; A \in X \quad (\text{membership axiom}, \mathcal{M}) \qquad \frac{X \twoheadrightarrow Y,\; Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \quad (\text{pseudo-transitivity}, \mathcal{T})$$

together with exactly one inference rule of

$$\frac{X \twoheadrightarrow Y,\; X \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \quad (\text{union}, \mathcal{U}) \qquad \frac{X \twoheadrightarrow Y,\; X \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z} \quad (\text{intersection}, \mathcal{I}) \qquad \frac{X \twoheadrightarrow Y,\; X \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \quad (\text{difference}, \mathcal{D})$$

form a minimal, sound and complete set of inference rules for the implication of MVDs in relational databases.

For the remainder of this section, we will be concerned with formally verifying Theorem 1.

3.1. Completeness

The soundness of each inference rule is obvious.

Lemma 2. $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}, \mathcal{U}\}$ *is complete for the implication of MVDs.*

Proof. It is sufficient to show that the augmentation rule $\mathcal{A}$ follows from $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}, \mathcal{U}\}$. The lemma follows then from the completeness of $\{\mathcal{C}.1, \mathcal{A}, \mathcal{T}\}$ [9].

We show first that the reflexivity axiom $\mathcal{R}$ follows from $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}, \mathcal{U}\}$.

Suppose $R = \emptyset$. In this case the only instance of $\mathcal{R}$ is $\emptyset \twoheadrightarrow \emptyset$ which is also an instance of the R-axiom. We can therefore assume that $R \neq \emptyset$ for the rest of the proof.

Suppose $X \neq \emptyset$. We proceed by induction on the number n of attributes in Y. If $n = 0$, then we have the following inference:

$$\begin{array}{l} \mathcal{M}: \overline{X \twoheadrightarrow A}^{\,A \in X} \quad \mathcal{M}: \overline{A \twoheadrightarrow A}^{\,A \in \{A\}} \\ \mathcal{T}: \hline \qquad\qquad X \twoheadrightarrow \emptyset \end{array}.$$

Suppose $Y = \{A_1, \ldots, A_n, A_{n+1}\}$. Note that $\{A_1, \ldots, A_n\} \subseteq X$ and $A_{n+1} \in X$ as $Y \subseteq X$. We then have the following inference:

$$\begin{array}{l} \mathcal{R}(\text{hypothesis}): \overline{X \twoheadrightarrow \{A_1, \ldots, A_n\}}^{\,\{A_1,\ldots,A_n\} \subseteq X} \quad \mathcal{M}: \overline{X \twoheadrightarrow A_{n+1}}^{\,A_{n+1} \in X} \\ \mathcal{U}: \hline \qquad\qquad X \twoheadrightarrow Y \end{array}.$$

It remains to consider the case where $X = \emptyset$. Note that $\frac{}{R \twoheadrightarrow R}\; R \subseteq R$ follows from the previous case as $R \neq \emptyset$.

$$\begin{array}{l} \mathcal{C}.1: \overline{\emptyset \twoheadrightarrow R} \quad \mathcal{R}(R \neq \emptyset): \overline{R \twoheadrightarrow R}^{\,R \subseteq R} \\ \mathcal{T}: \hline \qquad\qquad \emptyset \twoheadrightarrow \emptyset \end{array}.$$

This shows that the reflexivity axiom $\mathcal{R}$ follows from $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}, \mathcal{U}\}$.

The following inference shows that the augmentation rule $\mathcal{A}$ is derivable from $\{\mathcal{R}, \mathcal{T}, \mathcal{U}\}$. Note that $Y = (Y - X) \cup (Y \cap X)$.

$$\mathcal{U}: \dfrac{\mathcal{U}: \dfrac{\mathcal{T}: \dfrac{\mathcal{R}: \overline{XW \twoheadrightarrow X}^{X \subseteq XW} \qquad X \twoheadrightarrow Y}{XW \twoheadrightarrow Y - X} \qquad \mathcal{R}: \overline{XW \twoheadrightarrow Y \cap X}^{Y \cap X \subseteq XW}}{XW \twoheadrightarrow Y} \qquad \mathcal{R}: \overline{XW \twoheadrightarrow V}^{V \subseteq W \subseteq XW}}{XW \twoheadrightarrow YV}$$

This concludes the proof. □

Lemma 3. *The complementation rule* $\mathcal{C}$ *is derivable from* $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}\}$.

Proof. Consider the complementation rule $\mathcal{C}$ and suppose first that $Y = \emptyset$. In this case we can obtain $\mathcal{C}$ by the following inference.

$$\mathcal{T}: \dfrac{X \twoheadrightarrow \emptyset \qquad \mathcal{C}.1: \overline{\emptyset \twoheadrightarrow R}}{X \twoheadrightarrow R}.$$

Suppose now that $Y \neq \emptyset$. In this case we can obtain $\mathcal{C}$ by the following inference:

$$\mathcal{T}: \dfrac{X \twoheadrightarrow Y \qquad \mathcal{T}: \dfrac{\mathcal{T}: \dfrac{\mathcal{M}: \overline{Y \twoheadrightarrow A}^{A \in Y} \quad \mathcal{M}: \overline{A \twoheadrightarrow A}^{A \in \{A\}}}{Y \twoheadrightarrow \emptyset} \qquad \mathcal{C}.1: \overline{\emptyset \twoheadrightarrow R}}{Y \twoheadrightarrow R}}{X \twoheadrightarrow R - Y}.$$

This proves the lemma. □

Lemma 4. $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}, \mathcal{I}\}$ *is complete for the implication of MVDs.*

Proof. We show that the union rule $\mathcal{U}$ is derivable from $\{\mathcal{C}, \mathcal{I}\}$. The lemma follows, therefore, from Lemmas 3 and 2. Note that $Y \cup Z = R - ((R - Y) \cap (R - Z))$ by the de Morgan law.

$$\mathcal{C}: \dfrac{\mathcal{I}: \dfrac{\mathcal{C}: \dfrac{X \twoheadrightarrow Y}{X \twoheadrightarrow R - Y} \qquad \mathcal{C}: \dfrac{X \twoheadrightarrow Z}{X \twoheadrightarrow R - Z}}{X \twoheadrightarrow (R - Y) \cap (R - Z)}}{X \twoheadrightarrow R - ((R - Y) \cap (R - Z))}$$

This concludes the proof. □

Lemma 5. $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}, \mathcal{D}\}$ *is complete for the implication of MVDs.*

Proof. We show that the intersection rule $\mathcal{I}$ is derivable from $\{\mathcal{D}\}$. The lemma follows then from Lemma 4. Note that $Y \cap Z = Z - (Z - Y)$.

$$\mathcal{D}: \dfrac{\mathcal{D}: \dfrac{X \twoheadrightarrow Y \qquad X \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \qquad X \twoheadrightarrow Z}{X \twoheadrightarrow Z - (Z - Y)}$$

This concludes the proof. □

3.2. Minimality

We will now verify that in all three axiomatisations every rule is independent from the rest of the rules. That is, none of the inference rules can be omitted without losing completeness. The independence proofs have been computationally verified using GNU-Pascal programs. These programs determine the syntactic hull $\Sigma^+_{\mathfrak{S}}$ of Σ by recursively applying all inference rules in $\mathfrak{S}$ until no further MVDs are inferred by any of the rules. We also indicate for each independence proof why a certain MVD σ does not belong to $\Sigma^+_{\mathfrak{S}}$.

Lemma 6. *The R-axiom* $\mathcal{C}.1$ *is independent from the set* $\mathfrak{S} = \{\mathcal{M}, \mathcal{T}, \mathcal{U}, \mathcal{I}, \mathcal{D}\}$.

Proof. Let $R = \{A\}$, $\Sigma = \emptyset$ and $\sigma = \emptyset \twoheadrightarrow A$. We present $\Sigma^+_{\mathfrak{S}}$ by the following table. The MVD $X \twoheadrightarrow Y$ is in $\Sigma^+_{\mathfrak{S}}$ if and only if the entry in line X and column Y is a cross $\times$. The membership-axiom $\mathcal{M}$ introduces only MVDs with left-hand side A and the remaining rules in $\mathfrak{S}$ leave the left-hand sides invariant.

	$\emptyset$	A
$\emptyset$		
A	$\times$	$\times$

This shows that $\sigma \notin \Sigma^+_{\mathfrak{S}}$, but σ can be inferred using the R-axiom $\mathcal{C}.1$. □

Lemma 7. $\mathcal{M}$ *is independent from the set* $\mathfrak{S} = \{\mathcal{C}.1, \mathcal{T}, \mathcal{U}, \mathcal{I}, \mathcal{D}\}$.

Proof. Let $R = \{A\}$, $\Sigma = \emptyset$ and $\sigma = A \twoheadrightarrow A$. $\Sigma^+_{\mathfrak{S}}$ is presented by the following table. The R-axiom $\mathcal{C}.1$ introduces only $\emptyset \twoheadrightarrow A$ and the remaining rules in $\mathfrak{S}$ leave the left-hand sides invariant.

	$\emptyset$	A
$\emptyset$	$\times$	$\times$
A		

This shows that $\sigma \notin \Sigma^+_{\mathfrak{S}}$, but σ can be inferred using $\mathcal{M}$. □

Lemma 8. $\mathcal{T}$ *is independent from the set* $\mathfrak{S} = \{\mathcal{C}.1, \mathcal{M}, \mathcal{U}, \mathcal{I}, \mathcal{D}\}$.

Proof. Let $R = \{A, B\}$, $\Sigma = \emptyset$ and $\sigma = A \twoheadrightarrow AB$. $\Sigma^+_{\mathfrak{S}}$ is presented by the following table. The axioms $\mathcal{C}.1$ and $\mathcal{M}$ introduce some MVDs for every left hand side, the difference rule $\mathcal{D}$ generates the $\emptyset$-column and the union rule $\mathcal{U}$ introduces the new MVD $AB \twoheadrightarrow AB$. Further applications do not result in new MVDs.

	$\emptyset$	A	B	AB
$\emptyset$	$\times$			$\times$
A	$\times$	$\times$		
B	$\times$		$\times$	
AB	$\times$	$\times$	$\times$	$\times$

This shows that $\sigma \notin \Sigma^+_{\mathfrak{S}}$, but σ can be inferred by applying $\mathcal{T}$ to $A \twoheadrightarrow \emptyset, \emptyset \twoheadrightarrow AB \in \Sigma^+_{\mathfrak{S}}$. □

Lemma 9. $\mathcal{U}$ *is independent from* $\mathfrak{S} = \{\mathcal{C}.1, \mathcal{M}, \mathcal{T}\}$.

Proof. Let $R = \{A, B, C\}$, $\Sigma = \emptyset$ and $\sigma = AB \twoheadrightarrow AB$. $\Sigma^+_{\mathfrak{S}}$ is presented by the following table. Note that the complementation rule $\mathcal{C}$ follows from $\mathfrak{S}$ according to Lemma 3. The table may be generated by hand as follows. First the axioms $\mathcal{C}.1$ and $\mathcal{M}$ are applied. The complementation rule $\mathcal{C}$ generates the $\emptyset$-row. The pseudo-transitivity rule then generates the $\emptyset$-column. Finally, the complementation rule $\mathcal{C}$ generates all remaining rows. Further applications of the pseudo-transitivity rule result in MVDs that were already generated before.

	$\emptyset$	A	B	C	AB	AC	BC	ABC
$\emptyset$	×							×
A	×	×					×	×
B	×		×			×		×
C	×			×	×			×
AB	×	×	×			×	×	×
AC	×	×		×	×		×	×
BC	×		×	×	×	×		×
ABC	×	×	×	×	×	×	×	×

This shows that $\sigma \notin \Sigma^+_{\mathfrak{S}}$, but σ can be inferred applying $\mathcal{U}$ to $AB \twoheadrightarrow A$, $AB \twoheadrightarrow B \in \Sigma^+_{\mathfrak{S}}$. □

Lemma 10. *$\mathcal{I}$ is independent from* $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}\}$.

Proof. If $\mathcal{I}$ was not independent from $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}\}$, then $\mathcal{U}$ would not be independent from $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}\}$ according to the proof of Lemma 4. This, however, is a contradiction to Lemma 9. □

Lemma 11. *$\mathcal{D}$ is independent from* $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}\}$.

Proof. If $\mathcal{D}$ was not independent from $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}\}$, then $\mathcal{I}$ would not be independent from $\{\mathcal{C}.1, \mathcal{M}, \mathcal{T}\}$ according to the proof of Lemma 5. This, however, is a contradiction to Lemma 10. □

The previous lemmata verify Theorem 1. An interesting fact is that in each of the proofs we could find trivial MVDs as witnesses for the independence of the respective inference rules. This is an even stronger indication for the need of each inference rule.

4. Discussion

Each of the three axiomatisations of Theorem 1 provides a positive solution to Fagin's question. The power of the complementation rule $\mathcal{C}$ has been reduced to the simple R-axiom $\mathcal{C}.1$, and the powerful augmentation rule $\mathcal{A}$ has been reduced to the membership axiom $\mathcal{M}$. The loss of completeness is compensated by either one of the union rule $\mathcal{U}$, intersection rule $\mathcal{I}$ or difference rule $\mathcal{D}$. We believe that any of these three sets provides a simple syntactic description of the implication of multivalued dependencies.

The three axiomatisations add further new choices to the minimal sets of inference rules that have previously been proposed [23,9] to capture the implication of MVDs in relational databases.

Moreover, we can characterise syntactically the difference between functional and multivalued dependencies. Armstrong [2,3] gives the following sound, complete and minimal set of inference rules for the implication of FDs.

$$\frac{}{X \to Y}\ Y \subseteq X \qquad \frac{X \to Y}{X \to XY} \qquad \frac{X \to Y,\ Y \to Z}{X \to Z}$$

(reflexivity) (extension) (transitivity)

It is not difficult to show that the extension rule can be replaced by the union rule resulting in the following minimal axiomatisation for the implication of FDs.

$$\frac{}{X \to Y}\ Y \subseteq X \qquad \frac{X \to Y,\ X \to Z}{X \to YZ} \qquad \frac{X \to Y,\ Y \to Z}{X \to Z}$$

(reflexivity) (union) (transitivity)

The set consisting of $\mathcal{C}.1$, $\mathcal{M}$, $\mathcal{T}$ and $\mathcal{U}$ gives an axiomatisation of MVDs in which $\mathcal{M}$ may be replaced by the more powerful reflexivity axiom $\mathcal{R}$ (and still maintains minimality).

$$\frac{}{\emptyset \twoheadrightarrow R} \qquad \frac{}{X \twoheadrightarrow Y}\ Y \subseteq X \qquad \frac{X \twoheadrightarrow Y,\ X \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \qquad \frac{X \twoheadrightarrow Y,\ Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

(R-axiom) (reflexivity) (union) (pseudo-transitivity)

So, the difference between the axiomatisation of FDs and the axiomatisation of MVDs is that MVDs require the R-axiom, which reflects the dependence of the definition of MVDs on the underlying relation schema R, and that only pseudo-transitivity holds for MVDs whereas general transitivity holds for FDs.

Finally, the following set of inference rules

$$\frac{}{X \to Y}\ Y \subseteq X \qquad \frac{X \to Y}{X \to XY} \qquad \frac{X \to Y,\ Y \to Z}{X \to Z} \qquad \frac{}{\emptyset \twoheadrightarrow R}$$

$$\frac{X \twoheadrightarrow Y,\ Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \qquad \frac{X \to Y}{X \twoheadrightarrow Y} \qquad \frac{X \twoheadrightarrow Y,\ Y \to Z}{X \to Z - Y}$$

together with exactly one inference rule of

$$\frac{X \twoheadrightarrow Y,\ X \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \qquad \frac{X \twoheadrightarrow Y,\ X \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z} \qquad \frac{X \twoheadrightarrow Y,\ X \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

is sound and complete for the implication of both FDs and MVDs in relational databases. In each of these sets the complementation rule $\mathcal{C}$ is reduced to the R-axiom $\mathcal{C}.1$ and the augmentation rule $\mathcal{A}$ is dropped.

5. Conclusion and further work

We have studied the problem of capturing the implication of MVDs in relational databases by a set of inference rules in which every one of the rules is essential. Fagin asked whether there is any such set in which the power of the complementation rule $\mathcal{C}$ can be reduced and the augmentation rule $\mathcal{A}$ can be dropped at the same time. Biskup has shown that one can dispense with only one of the rules at the same time if one considers $\{\mathcal{R}, \mathcal{C}.1, \mathcal{C}, \mathcal{A}, \mathcal{T}\}$.

This paper shows that an affirmative answer to Fagin's problem can still be achieved if one considers further inference rules for MVDs. The major results are three minimal, sound and complete sets of inference rules for the implication of MVDs in relational databases in each of which the complementation rule $\mathcal{C}$ is reduced to the R-axiom $\mathcal{C}.1$ and the augmentation rule $\mathcal{A}$ is reduced to the membership axiom $\mathcal{M}$. In order to regain completeness exactly one of union rule $\mathcal{U}$, intersection rule $\mathcal{I}$ or difference rule $\mathcal{D}$ is sufficient.

Biskup [10] introduces MVDs in a context where the underlying set of attributes is left undetermined, and a sound and complete set $\mathfrak{S}_0$ of inference rules for the implication of such MVDs was obtained. The augmentation rule is part of $\mathfrak{S}_0$ [10, p. 104], and it would be interesting to see whether the results from our article can provide alternative axiomatisations for this notion of an MVD.

Finally, our result may simplify the quest of finding suitable and comprehensible extensions for the notion of an MVD in any advanced data models.

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