On Multivalued Dependencies in fixed and undetermined Universes

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3. Axiomatisations in fixed Universes
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1.1 Functional Dependencies

- IRD-No $\longrightarrow$ Name and Name $\longrightarrow$ E-mail imply IRD-No $\longrightarrow$ E-mail
- that is independent from the universe the first two FDs live in
- {IRD-No,Name,E-mail}

IRD-No	Name	E-mail
123	Bugs Bunny	no.ears@hotbunny.com
234	Daffy Duck	no.carrots@tasty.org
345	Roadrunner	no.tyres@dash.ac.lt

- {IRD-No,Name,E-mail,Status}

IRD-No	Name	E-mail	Status
123	Bugs Bunny	no.ears@hotbunny.com	Single
234	Daffy Duck	no.carrots@tasty.org	Single
345	Roadrunner	no.tyres@dash.ac.lt	Married

1.2 Multivalued Dependencies

- Employee $\twoheadrightarrow$ Child
- Is Employee $\twoheadrightarrow$ Salary implied or not implied?

Employee	Child	Salary
Homer	Bart	4000
Homer	Lisa	5000
Homer	Bart	5000
Homer	Lisa	4000

Employee	Child	Salary	Year
Homer	Bart	4000	2004
Homer	Lisa	5000	2005
Homer	Bart	5000	2005
Homer	Lisa	4000	2004

- Employee $\twoheadrightarrow$ Salary not a consequence of Employee $\twoheadrightarrow$ Child
- only reflects normalisation on planet {Employee, Child, Salary}

1.3. The Standard Notion of R-implication

- $\Sigma = \{X_1 \twoheadrightarrow Y_1, \ldots, X_k \twoheadrightarrow Y_k\}$, $X \twoheadrightarrow Y$ on R $(X \cup Y \cup \bigcup_{i=1}^{k} (X_i \cup Y_i) \subseteq R)$

  Σ R-implies $X \twoheadrightarrow Y$ iff $\forall r \subseteq dom(R)$: if $\models_r \Sigma$, then $\models_r X \twoheadrightarrow Y$

$$\frac{}{X \twoheadrightarrow Y}\, Y \subseteq X$$
(reflexivity, $\mathcal{R}$)

$$\frac{X \twoheadrightarrow Y}{XU \twoheadrightarrow YV}\, V \subseteq U$$
(augmentation, $\mathcal{A}$)

$$\frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$
(pseudo-transitivity, $\mathcal{T}$)

$$\frac{X \twoheadrightarrow Y}{X \twoheadrightarrow R - Y}$$
(R-complementation, $\mathcal{C}_R$)

$$\frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow YZ}$$
(union, $\mathcal{U}$)

$$\frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$
(difference, $\mathcal{D}$)

$$\frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z}$$
(intersection, $\mathcal{I}$)

- $R\mathfrak{F} = \langle \mathcal{R}, \mathcal{A}, \mathcal{T}, \mathcal{C}_R \rangle$ R-sound, R-complete ($\forall \Sigma$ on R: $\Sigma^+_{R\mathfrak{F}} = \Sigma^*_R$)

1.4. Complementary Axiomatisations

- $R\mathfrak{S} = \mathfrak{S} \cup \{\mathcal{C}_R\}$ said to be R-*complementary*:

  every inference of $X \twoheadrightarrow Y$ using $R\mathfrak{S}$ can be turned into inference of $X \twoheadrightarrow Y$ using $R\mathfrak{S}$ such that R-complementation rule $\mathcal{C}_R$ applied at most once and if, then in the last step

- formally:

$$X \twoheadrightarrow Y \in \Sigma^+_{R\mathfrak{S}} \quad \text{iff} \quad X \twoheadrightarrow Y \in \Sigma^+_{\mathfrak{S}} \text{ or } X \twoheadrightarrow (R-Y) \in \Sigma^+_{\mathfrak{S}}$$

  where $\Sigma = \{X_1 \twoheadrightarrow Y_1, \ldots, X_k \twoheadrightarrow Y_k\}$ and $X \cup Y \cup \bigcup_{i=1}^{k} (X_i \cup Y_i) \subseteq R$

- $R\mathfrak{S}$ said to be *sound* (*complete*, *complementary*) iff $R\mathfrak{S}$ is R-sound (R-complete, R-complementary) for all R
- $R\mathfrak{F}$ from *Beeri,Fagin,Howard (1977)* not complementary

1.5. Why $R\mathfrak{F}$ isn't complementary

- $\Sigma = \{\text{Movie} \twoheadrightarrow \text{Actor}, \text{Movie} \twoheadrightarrow \text{Feature}\}$
- $\text{Movie} \twoheadrightarrow \text{Actor, Feature} \notin \Sigma^+_{\{\mathcal{R},\mathcal{A},\mathcal{T}\}}$
- $\text{Movie} \twoheadrightarrow Y \notin \Sigma^+_{\{\mathcal{R},\mathcal{A},\mathcal{T}\}}$ $\forall Y$ with $Y - \{\text{Movie,Actor,Feature}\} \neq \emptyset$
- for $R := \{\text{Movie, Actor, Yearborn, Feature}\}$ we have
$$\text{Movie} \twoheadrightarrow \text{Actor, Feature} \in \Sigma^+_{R\mathfrak{F}}$$
- in any such inference $\mathcal{C}_R$ must be used at least once, but $R - \{\text{Actor,Feature}\} = \{\text{Movie,Yearborn}\}$ implies that $\mathcal{C}_R$ is not just used as last rule

1.6. Example Derivation

- A = Movie, B = Actor, C = Title, D = YearBorn
- inference of $A \twoheadrightarrow B, C$ from $\Sigma = \{A \twoheadrightarrow B, A \twoheadrightarrow C\}$ using $\langle \mathcal{R}, \mathcal{A}, \mathcal{T}, \mathcal{C}_R \rangle$

$$\cfrac{\cfrac{\cfrac{\cfrac{A \twoheadrightarrow B}{A \twoheadrightarrow A, B}\,\mathcal{A} \qquad \cfrac{\cfrac{A \twoheadrightarrow C}{A \twoheadrightarrow A, B, D}\,\mathcal{C}_R}{A, B \twoheadrightarrow A, B, D}\,\mathcal{A}}{A \twoheadrightarrow D}\,\mathcal{T}}{A \twoheadrightarrow A, D}\,\mathcal{A}}{A \twoheadrightarrow B, C}\,\mathcal{C}_R$$

1.7. Axiomatisations in fixed Universes

- are there complementary axiomatisations at all?

$$\frac{}{\emptyset \twoheadrightarrow \emptyset} \quad \text{(empty-set-axiom,}\mathcal{R}_\emptyset) \qquad \frac{X \twoheadrightarrow Y}{XU \twoheadrightarrow YV}\; V \subseteq U \quad (\mathcal{A}) \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \quad (\mathcal{T})$$

$$\frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \quad \text{(additive transitivity, } \mathcal{T}^*) \qquad \frac{X \twoheadrightarrow Y, W \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z}\; Y \cap W = \emptyset \quad \text{(subset, } \mathcal{S})$$

- Biskup (1980):
  $R\mathfrak{B}_0 = \langle \mathcal{R}_\emptyset, \mathcal{A}, \mathcal{T}, \mathcal{T}^*, \mathcal{S}, \mathcal{C}_R \rangle$ complete + complementary
- **Objective 1:** Find all complete and complementary subsets of

$$R\mathfrak{S}_U = \{\mathcal{R}, \mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{T}^*, \mathcal{U}, \mathcal{D}, \mathcal{I}, \mathcal{C}_R\}$$

1.8. Example Derivation

- A = Movie, B = Actor, C = Title, D = YearBorn
- inference of $A \twoheadrightarrow B, C$ from $\Sigma = \{A \twoheadrightarrow B, A \twoheadrightarrow C\}$ using $\langle \mathcal{R}_\emptyset, \mathcal{A}, \mathcal{T}, \mathcal{T}^*, \mathcal{S}, \mathcal{C}_R \rangle$

$$
\dfrac{\dfrac{\dfrac{}{\emptyset \twoheadrightarrow \emptyset}\,\mathcal{R}_\emptyset}{A \twoheadrightarrow A}\,\mathcal{A} \qquad \dfrac{\dfrac{A \twoheadrightarrow B}{A \twoheadrightarrow A, B}\,\mathcal{A} \qquad \dfrac{A \twoheadrightarrow C}{A, B \twoheadrightarrow C}\,\mathcal{A}}{A \twoheadrightarrow A, B, C}\,\mathcal{T}^*}{A \twoheadrightarrow B, C}\,\mathcal{T}
$$

1.9. Implication in undetermined Universes

- consequences dependent on the universe are in fact no consequences
- expression: $X \twoheadrightarrow Y$ with finite $X, Y \subseteq \mathfrak{A}$
- $\models_r X \twoheadrightarrow Y$ iff $X \cup Y \subseteq Dom(r)$ and $r = r[XY] \bowtie r[X \cup (Dom(r) - Y)]$
- $\Sigma = \{X_1 \twoheadrightarrow Y_1, \ldots, X_k \twoheadrightarrow Y_k\} \models X \twoheadrightarrow Y$ iff
  for each relation r with $X \cup Y \cup \bigcup_{i=1}^{k} (X_i \cup Y_i) \subseteq Dom(r)$ we have
  $\models_r X \twoheadrightarrow Y$ whenever $\models_r \Sigma$
- $X \cup Y \cup \bigcup_{i=1}^{k} (X_i \cup Y_i) \subseteq R$:
  Σ R-implies $X \twoheadrightarrow Y$ whenever Σ implies $X \twoheadrightarrow Y$, but not vice versa!

1.10. Capturing Implication in undetermined Universes

- Biskup (1980):

$$\frac{}{\emptyset \twoheadrightarrow \emptyset} \quad \frac{X \twoheadrightarrow Y}{XU \twoheadrightarrow YV}\, V \subseteq U \quad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

(empty-set-axiom, $\mathcal{R}_\emptyset$) $(\mathcal{A})$ $(\mathcal{T})$

$$\frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \quad \frac{X \twoheadrightarrow Y, W \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z}\, Y \cap W = \emptyset$$

(additive transitivity, $\mathcal{T}^*$) (subset, $\mathcal{S}$)

- $\mathfrak{B}_0$ is sound and complete (for all finite Σ we have $\Sigma^+_{\mathfrak{B}_0} = \Sigma^*$)
- **Objective 2:** Find all minimal complete subsets of

$$\mathfrak{S}_U = \{\mathcal{R}, \mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{T}^*, \mathcal{U}, \mathcal{D}, \mathcal{I}\}$$

- **Objective 3:** Determine the implication problem's time-complexity!

2.1. Minimality in undetermined Universes

- complete set $\mathfrak{S}$ *minimal* iff for all $\mathfrak{R} \in \mathfrak{S}$: $\mathfrak{S} - \{\mathfrak{R}\}$ is incomplete
- $\mathfrak{R}$ *independent* from $\mathfrak{S}$ iff $\Sigma^+_{\mathfrak{S}} \subset \Sigma^+_{\mathfrak{S} \cup \{\mathfrak{R}\}}$ for some finite Σ
- $\mathfrak{S}$ minimal iff for all $\mathfrak{R} \in \mathfrak{S}$: $\mathfrak{R}$ independent from $\mathfrak{S} - \{\mathfrak{R}\}$
- $\mathcal{R}$ is independent from $\{\mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{T}^*, \mathcal{U}, \mathcal{D}, \mathcal{I}\}$
  - $\Sigma = \emptyset$ and $\sigma = \emptyset \twoheadrightarrow \emptyset$,
  - $\sigma \notin \Sigma^+_{\mathfrak{S}}$,
  - but $\sigma \in \Sigma^+_{\mathfrak{S} \cup \{\mathcal{R}\}}$

2.2. Another Independence Proof

- $\mathcal{S}$ is independent from $\{\mathcal{R}, \mathcal{A}, \mathcal{T}, \mathcal{T}^*, \mathcal{U}, \mathcal{D}, \mathcal{I}\}$
  - $\Sigma = \{A \twoheadrightarrow BC, D \twoheadrightarrow CD\}$, and $\sigma = A \twoheadrightarrow C$

	$\emptyset$	A	B	C	D	AB	AC	AD	BC	BD	CD	ABC	ABD	ACD	BCD	$ABCD$
$\emptyset$	×															
A	×	×							×			×				
B	×		×													
C	×			×												
D	×			×	×						×					
AB	×	×	×	×		×	×		×			×				
AC	×	×	×	×		×	×		×			×				
AD	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
BC	×		×	×					×							
BD	×		×	×	×				×	×	×				×	
CD	×			×	×						×					
ABC	×	×	×	×		×	×		×			×				
ABD	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
ACD	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
BCD	×		×	×	×				×	×	×				×	
$ABCD$	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×

2.3 Some Derivations

- $\mathcal{A}$ is derivable from $\{\mathcal{R}, \mathcal{T}, \mathcal{U}\}$

$$\dfrac{\dfrac{\dfrac{}{XU \twoheadrightarrow X}X\subseteq XU \quad X \twoheadrightarrow Y}{XU \twoheadrightarrow Y - X} \quad \dfrac{}{XU \twoheadrightarrow Y \cap X}Y\cap X\subseteq XU}{XU \twoheadrightarrow Y} \qquad \dfrac{}{XU \twoheadrightarrow V}V\subseteq U\subseteq XU$$

$$\dfrac{XU \twoheadrightarrow Y \qquad XU \twoheadrightarrow V}{XU \twoheadrightarrow YV}$$

- $\mathcal{T}^*$ is derivable from $\{\mathcal{T}, \mathcal{U}\}$

$$\dfrac{\dfrac{X \twoheadrightarrow Y \quad Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \quad X \twoheadrightarrow Y}{X \twoheadrightarrow Y \cup Z}$$

2.4 The Set $\mathfrak{L}_1$

- recall that $\mathfrak{B}_0 = \langle \mathcal{R}_\emptyset, \mathcal{A}, \mathcal{T}, \mathcal{T}^*, \mathcal{S} \rangle$ is complete
- subset $\mathfrak{B} = \langle \mathcal{R}, \mathcal{A}, \mathcal{T}, \mathcal{T}^*, \mathcal{S} \rangle$ of $\mathfrak{S}_U$ also complete
- Theorem:
  $\mathfrak{L}_1 = \langle \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U} \rangle$ is complete for the implication of MVDs.

  - empty-set-axiom $\mathcal{R}_\emptyset$,
  - augmentation rule $\mathcal{A}$ and
  - additive transitivity rule $\mathcal{T}^*$ are all derivable from $\mathfrak{L}_1$

2.5 More Derivations

- $\mathcal{T}$ is derivable from $\{\mathcal{T}^*, \mathcal{D}\}$

$$\frac{X \twoheadrightarrow Y \qquad \dfrac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Y \cup Z}}{X \twoheadrightarrow \underbrace{(Y \cup Z) - Y}_{=Z-Y}}$$

- $\mathcal{U}$ is derivable from $\{\mathcal{R}, \mathcal{T}^*, \mathcal{D}\}$

$$\frac{\dfrac{\dfrac{\dfrac{}{X \twoheadrightarrow X} \quad X \twoheadrightarrow Y}{X \twoheadrightarrow XY} \qquad \dfrac{\dfrac{}{XY \twoheadrightarrow X} \quad X \twoheadrightarrow Z}{XY \twoheadrightarrow XZ}}{X \twoheadrightarrow XYZ} \qquad \dfrac{}{X \twoheadrightarrow (X - (YZ))}}{X \twoheadrightarrow \underbrace{XYZ - (X - (YZ))}_{=YZ}}$$

2.6 All minimal Sets in undetermined Universes

- Theorem:
  $\mathfrak{L}_2 = \langle \mathcal{R}, \mathcal{S}, \mathcal{T}^*, \mathcal{D} \rangle$ is complete for the implication of MVDs
- Theorem (Objective 2):
  The only minimal complete subsets of $\mathfrak{S}_U$ for the implication of MVDs are $\mathfrak{B}$, $\mathfrak{L}_1$ and $\mathfrak{L}_2$.
- consider every $\mathfrak{S}$ of $\mathfrak{S}_U$ including $\mathcal{R}$ and $\mathcal{S}$,
  if $\mathfrak{S}$ is not a superset of $\mathfrak{B}$, $\mathfrak{L}_1$ or $\mathfrak{L}_2$, then at least one of
  - $\mathcal{U}$ is independent from $\{\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{T}^*, \mathcal{I}\}$
  - $\mathcal{T}$ is independent from $\{\mathcal{R}, \mathcal{S}, \mathcal{A}, \mathcal{T}^*, \mathcal{U}, \mathcal{I}\}$
  - $\mathcal{T}$ is independent from $\{\mathcal{R}, \mathcal{S}, \mathcal{A}, \mathcal{U}, \mathcal{D}, \mathcal{I}\}$

  shows that there is some rule in $\mathfrak{S}_U - \mathfrak{S}$ independent from $\mathfrak{S}$

3.1 Axiomatisations in fixed Universes

- $\mathfrak{R}$ is *R-independent* from $R\mathfrak{S}$ iff there is some Σ on R and some σ on R such that $\sigma \notin \Sigma^+_{R\mathfrak{S}}$ but $\sigma \in \Sigma^+_{R\mathfrak{S}\cup\{\mathfrak{R}\}}$
- $\mathfrak{R}$ *independent* from $R\mathfrak{S}$ iff there is some R such that $\mathfrak{R}$ is R-independent from $R\mathfrak{S}$
- $R\mathfrak{S}$ *minimal* iff for all $\mathfrak{R} \in R\mathfrak{S}$: $\mathfrak{R}$ independent from $R\mathfrak{S} - \{\mathfrak{R}\}$
- Theorem:
  Let $\mathfrak{S}$ be a sound set of inference rules for the implication of MVDs. The set $\mathfrak{S}$ is complete for the implication of MVDs if and only if $R\mathfrak{S}$ is complete and complementary for the R-implication of MVDs.
- Corollary:
  $R\mathfrak{L}_1$ and $R\mathfrak{L}_2$ are sound, complete and complementary for the R-implication of MVDs.

3.2 Minimality in fixed Universes

- Theorem:
  The complete sets $R\mathfrak{B}$, $R\mathfrak{L}_1$, and $R\mathfrak{L}_2$ are not minimal.

- Theorem:
  Let $\mathfrak{S}$ be a sound set of inference rules for the implication of MVDs. The set $\mathfrak{S}$ is minimal and complete for the implication of MVDs if and only if $R\mathfrak{S}$ is complete and complementary for the R-implication of MVDs, and there is no inference rule $\mathfrak{R} \in \mathfrak{S}$ such that the set $R(\mathfrak{S} - \{\mathfrak{R}\})$ is still both complete and complementary for the R-implication of MVDs.

- Corollary (Objective 1):
  There are no proper subsets of $R\mathfrak{B}, R\mathfrak{L}_1$ and $R\mathfrak{L}_2$ which are both complete and complementary for the R-implication of MVDs.

4.1 The R-Implication Problem

- decide whether Σ R-implies σ
- fundamental: dependency basis for $X \subseteq R$ wrt a set Σ of MVDs
- $Dep_R(X) = \{Y \mid X \twoheadrightarrow Y \in \Sigma^+_{R\mathfrak{B}_0}\}$
- $(Dep_R(X), \subseteq, \cup, \cap, -, \emptyset, R)$ is a finite Boolean powerset algebra
- $a \in P$ of poset $(P, \sqsubseteq, 0)$ with least element 0 is called an *atom* of $(P, \sqsubseteq, 0)$ iff every element $b \in P$ with $b \sqsubseteq a$ satisfies $b = 0$ or $b = a$
- $(P, \sqsubseteq, 0)$ is called *atomic* iff for all $b \in P$ with $b \neq 0$ there is atom $a \in P$ with $a \sqsubseteq b$
- every finite Boolean algebra is atomic

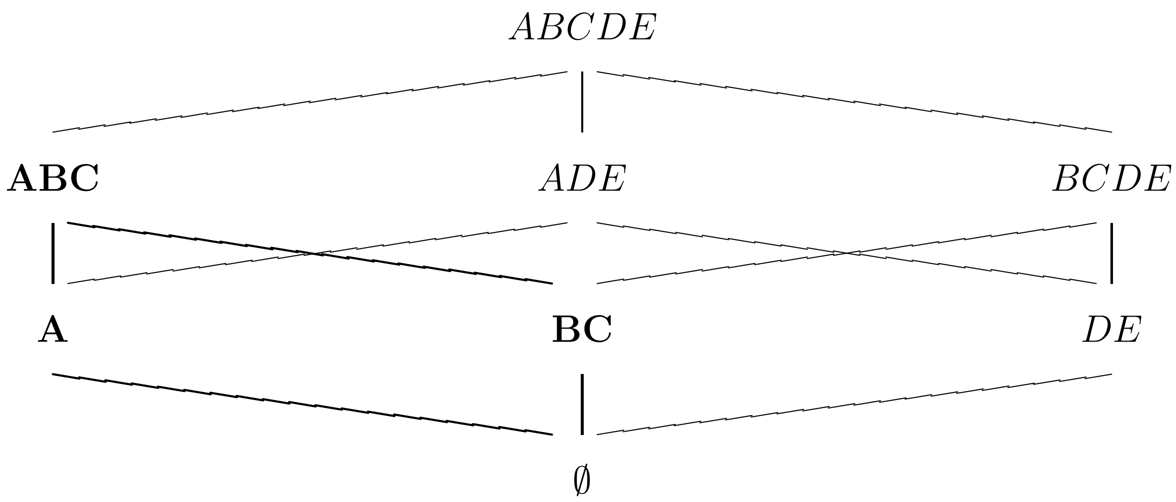
- dependency basis $DepB_R(X)$ is set of all atoms of $(Dep_R(X), \subseteq, \emptyset)$
- $X \twoheadrightarrow Y$ is R-implied by Σ iff $Y = \bigcup \mathcal{Y}$ for $\mathcal{Y} \subseteq DepB_R(X)$
- Theorem (Galil,1982):
  $DepB_R(X)$ can be computed in time $\mathcal{O}((1+\min\{s, \log \bar{p}\})\cdot n)$ where s denotes number of elements in Σ, $\bar{p}$ number of elements in $DepB_R(X)$ and n denotes the total number of occurrences of attributes in Σ
- Theorem (Galil,1982):
  $\Sigma \models X \twoheadrightarrow Y$ can be solved in time $\mathcal{O}((1+\min\{s, \log p\}) \cdot n)$ where s denotes number of elements in Σ, p number of elements in $DepB_R(X)$ that have non-empty intersection with Y and n denotes the total number of occurrences of attributes in Σ

4.2 The Implication Problem

- decide whether an arbitrary finite set Σ implies σ
- $Dep_U(X) = \{Y \mid X \twoheadrightarrow Y \in \Sigma^+_{\mathfrak{B}_0}\}$
- $(Dep_U(X), \subseteq, \cup, \cap, -, \emptyset, X^S)$ is finite Boolean algebra
- $DepB_U(X)$ of $X \subseteq \mathfrak{A}$ wrt Σ is set of all atoms of $(Dep_U(X), \subseteq, \emptyset)$
- $X^S = \bigcup Dep_U(X)$ is called the Σ-scope of X
- Theorem:
  $\Sigma \models X \twoheadrightarrow Y$ iff $Y = \bigcup \mathcal{Y}$ for some $\mathcal{Y} \subseteq DepB_U(X)$

4.3 An Example

- $\Sigma = \{A \twoheadrightarrow BC\}$ with $R = \{A, B, C, D, E\}$
- $DepB_R(A) = \{A, BC, DE\}$ are simply the atoms of the algebra
- $A^S = ABC$ and thus $DepB_U(A) = \{A, BC\}$



4.4 Relating $DepB_R(X)$ and $DepB_U(X)$

- $\Sigma = \{X_1 \twoheadrightarrow Y_1, \dots, X_k \twoheadrightarrow Y_k\}$
- $R_{\text{min}} := X \cup \bigcup_{i=1}^{k} (X_i \cup Y_i)$
- any algorithm computing $DepB_R(X)$ for any R with $R_{\text{min}} \subseteq R$ can be used to compute $DepB_U(X)$
- Theorem:
  Let Σ be a finite set of MVDs, $X \subseteq \mathfrak{A}$ some attribute set, and R some relation schema with $R_{\text{min}} \subseteq R$. Then

$$DepB_U(X) = \begin{cases} DepB_R(X) & \text{, if } X^S = R \\ DepB_R(X) - \{R - X^S\} & \text{, if } X^S \subset R \end{cases} .$$

4.5 Computing $DepB_U(X)$

Algorithm 1 (Dependency Basis)

Input: $\boldsymbol{\Sigma = \{X_1 \twoheadrightarrow Y_1, \ldots, X_k \twoheadrightarrow Y_k\}}$, and a set $\boldsymbol{X}$ of attributes
Output: $\boldsymbol{DepB_U(X)}$ with respect to $\boldsymbol{\Sigma}$

VAR $R_{\text{min}}, X^S_{\text{new}}, X^S_{\text{old}}, X^S_{\text{alg}}$: Set of attributes; *MVDList*: List of MVDs;

(1) $R_{\text{min}} := X \cup \bigcup_{i=1}^{k} (X_i \cup Y_i)$;
(2) Use the Algorithm from *Galil'1982* to compute $DepB_{R_{\text{min}}}(X)$;
(3) $X^S_{\text{new}} := X$;
(4) MVDList:= List of MVDs in Σ;
(5) REPEAT
(6) $\quad X^S_{\text{old}} := X^S_{\text{new}}$;
(7) $\quad$ Remove all attributes in X^S_{new} from the LHS of all MVDs in MVDList;
(8) $\quad$ FOR all MVDs $\emptyset \twoheadrightarrow Y$ in MVDList LET $X^S_{\text{new}} := X^S_{\text{new}} \cup Y$;
(9) UNTIL $X^S_{\text{new}} = X^S_{\text{old}}$;
(10) $X^S_{\text{alg}} := X^S_{\text{new}}$;
(11) IF $X^S_{\text{alg}} = R_{\text{min}}$ THEN RETURN($DepB_{R_{\text{min}}}(X)$)
(12) $\qquad$ ELSE RETURN($DepB_{R_{\text{min}}}(X) - \{R_{\text{min}} - X^S\}$);

□

4.6 Correctness and Time-Complexity of Computation

- Theorem:
  Algorithm 1 computes $DepB_U(X)$ with respect to Σ in time $\mathcal{O}((1 + \min\{s, \log \bar{p}\}) \cdot n)$ where s denotes the number of dependencies in Σ, $\bar{p}$ the number of sets in $DepB_U(X)$ and n denotes the total number of occurrences of attributes in Σ.

- correctness proof is essentially verification of computing the Σ-scope
  - Σ-scope X^S of X similar to attribute closure X^+ under set of FDs
  - use algorithm linear in total number of occurrences of attributes

- time bound follows from *Galil'1982*

4.7 Another Example

- $\Sigma = \{AB \twoheadrightarrow DEFG, CGJ \twoheadrightarrow ADHI\}$
- determine $DepB_U(\{A, C, G, J\})$ wrt Σ
- $R_{\text{min}} = \{A, B, C, D, E, F, G, H, I, J\}$
- $DepB_{R_{\text{min}}}(\{A, C, G, J\}) = \{\{A\}, \{C\}, \{G\}, \{J\}, \{D\}, \{H, I\}, \{B, E, F\}\}$
- Σ-scope of $\{A, C, G, J\}$ is $\{A, C, G, J\}^S = \{A, C, D, G, I, H, J\}$
- $\{A, C, G, J\}^S \subset R_{\text{min}}$ and $R_{\text{min}} - \{A, C, G, J\}^S = \{B, E, F\}$
- $DepB_U(\{A, C, G, J\}) = \{\{A\}, \{C\}, \{G\}, \{J\}, \{D\}, \{H, I\}\}$
- $ACGJ \twoheadrightarrow BDEF$ is R_{min}-implied by Σ, but not implied by Σ.

4.8 Deciding the Implication Problem

- Theorem:
  Let Σ be a finite set of MVDs. The MVD $X \twoheadrightarrow Y$ is implied by Σ if and only if $Y \subseteq X^S$ and $X \twoheadrightarrow Y$ is $R_{\min}$-implied by Σ.

- Theorem (Objective 3):
  The implication problem $\Sigma \models X \twoheadrightarrow Y$ can be decided in time $\mathcal{O}((1 + \min\{s, \log p\}) \cdot n)$ where s denotes the number of dependencies in Σ, p the number of sets in $DepB_U(X)$ that have non-empty intersection with Y and n denotes the total number of occurrences of attributes in Σ.

- there is a linear-time algorithm for computing the dependency basis in a fixed universe iff there is a linear-time algorithm for computing the dependency basis in undetermined universes

5.1 Axiomatisations for R-Implication

- Mendelzon'80: minimal, not complementary

$$\frac{}{X \twoheadrightarrow Y}\, Y \subseteq X \qquad \frac{X \twoheadrightarrow Y}{X \twoheadrightarrow R - Y} \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

- Biskup'80: minimal wrt both completeness and complementarity

$$\frac{}{X \twoheadrightarrow Y}\, Y \subseteq X \qquad \frac{X \twoheadrightarrow Y}{XU \twoheadrightarrow YV}\, V \subseteq U \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

$$\frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \qquad \frac{X \twoheadrightarrow Y, W \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z}\, Y \cap W = \emptyset \qquad \frac{X \twoheadrightarrow Y}{X \twoheadrightarrow R - Y}$$

5.1 Axiomatisations for R-Implication continued

- Link'06: minimal wrt both completeness and complementarity

$$\frac{}{X \twoheadrightarrow Y} \; Y \subseteq X \qquad \frac{X \twoheadrightarrow Y, W \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z} \; Y \cap W = \emptyset \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

$$\frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \qquad \frac{X \twoheadrightarrow Y}{X \twoheadrightarrow R - Y}$$

- Link'06: minimal wrt both completeness and complementarity

$$\frac{}{X \twoheadrightarrow Y} \; Y \subseteq X \qquad \frac{X \twoheadrightarrow Y, W \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z} \; Y \cap W = \emptyset \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow YZ}$$

$$\frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \qquad \frac{X \twoheadrightarrow Y}{X \twoheadrightarrow R - Y}$$

5.1 Axiomatisations for R-Implication continued

- Biskup'80: strongly minimal

$$\frac{X \twoheadrightarrow Y}{XU \twoheadrightarrow YV}\, V \subseteq U \qquad \frac{}{\emptyset \twoheadrightarrow R} \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

- Link/Hartmann'06: strongly minimal

$$\frac{}{X \twoheadrightarrow A}\, A \in X \qquad \frac{}{\emptyset \twoheadrightarrow R} \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

and exactly one of

$$\frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \qquad \frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z} \qquad \frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

5.2 Axiomatisations for Implication (almost R-complete)

- Biskup'80: minimal

$$\frac{}{X \twoheadrightarrow Y}\, Y \subseteq X \qquad \frac{X \twoheadrightarrow Y}{XU \twoheadrightarrow YV}\, V \subseteq U \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

$$\frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \qquad \frac{X \twoheadrightarrow Y, W \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z}\, Y \cap W = \emptyset$$

- Link'06: minimal

$$\frac{}{X \twoheadrightarrow Y}\, Y \subseteq X \qquad \frac{X \twoheadrightarrow Y, W \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z}\, Y \cap W = \emptyset \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \qquad \frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow YZ}$$

- Link'06: minimal

$$\frac{}{X \twoheadrightarrow Y}\, Y \subseteq X \qquad \frac{X \twoheadrightarrow Y, W \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z}\, Y \cap W = \emptyset \qquad \frac{X \twoheadrightarrow Y, Y \twoheadrightarrow Z}{X \twoheadrightarrow YZ} \qquad \frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y}$$

6.1 NMVDs (undefined, inapplicable, nonexistent)

- partial r satisfies NMVD $X \twoheadrightarrow Y$ iff for all $t_1, t_2 \in r$: if t_1, t_2 are X-total and $t_1[X] = t_2[X]$, then there is some $t \in r$ with $t[XY] = t_1[XY]$ and $t[X(R-Y)] = t_2[X(R-Y)]$
- Lien'82: $R\mathfrak{K} = \{\mathcal{R}, \mathcal{A}, \mathcal{U}, \mathcal{C}_R\}$ sound and complete for R-implication
- Link'06: sound and complete for R-implication is

$$\frac{}{\emptyset \twoheadrightarrow R} \quad (R\text{-axiom}, \mathcal{C}.1) \qquad \frac{}{A \twoheadrightarrow A} \quad (\text{attribute-axiom}, \mathcal{A}t)$$

$$\frac{X \twoheadrightarrow Y}{XA \twoheadrightarrow Y} \quad (\text{weak augmentation rule}, \mathcal{W}) \qquad \frac{X \twoheadrightarrow Y, X \twoheadrightarrow Z}{X \twoheadrightarrow Z - Y} \quad (\text{difference rule}, \mathcal{D})$$

6.2 NMVDs in fixed and undetermined Universes

- $R\mathfrak{K} = \{\mathcal{R}, \mathcal{A}, \mathcal{U}, \mathcal{C}_R\}$ is NOT complementary for R-implication
- Link'06:
  $R\mathfrak{L} = \{\mathcal{R}, \mathcal{A}, \mathcal{U}, \mathcal{D}, \mathcal{C}_R\}$ is complementary
- partial r satisfies NMVD $X \twoheadrightarrow Y$ iff $XY \subseteq Dom(r)$ and $r_X[Dom(r)] = r_X[XY] \bowtie r_X[X(Dom(r) - Y)]$
- Link'06:
  $\mathfrak{L} = \langle \mathcal{R}, \mathcal{A}, \mathcal{U}, \mathcal{D} \rangle$ only minimal, sound, complete subset for implication of NMVDs of $\{\mathcal{R}, \mathcal{A}, \mathcal{U}, \mathcal{D}, \mathcal{I}\}$

7 Interesting Things to look at

(i) Are there any complete sets (not a subset of $\mathfrak{S}_U$) in which the subset rule $\mathcal{S}$ does not occur?

(ii) Are there any minimal sets of inference rules that are also complementary?

(iii) Further investigate the notion of *strong minimality*, e.g. strongly minimal axiomatisations for FDs, MVDs, NFDs, NMVDs.

(iv) Consider complete axiomatisations of MVDs in Entity-Relationship Models. Are there complete axiomatisations over undetermined universes?

(v) Consider complete axiomatisations of MVDs in Nested Database Models. Are these complementary?

(vi) Consider complete axiomatisations of fuzzy and approximate MVDs. Are these complementary?